MEASURE AND INTEGRATION – FINAL EXAM SOLUTIONS
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1. (a) Let Ω be a set and $\mathcal{E}$ be a collection of subsets of Ω. Assume that we have sets $A_0 \subset A \subset \Omega$ such that $A_0 \neq A$ and

$$\text{for all } B \in \mathcal{E}, \text{ either } A \subset B \text{ or } A \cap B = \varnothing.$$

Prove that $A_0 \notin \sigma(\mathcal{E})$, where $\sigma(\mathcal{E})$ denotes the σ-algebra generated by $\mathcal{E}$.

Solution. Let $\mathcal{F} = \{B \subset \Omega : A \subset B \text{ or } A \cap B = \varnothing\}$. Let us show that $\mathcal{F}$ is a σ-algebra:

- We have $\Omega \in \mathcal{F}$ since $A \subset \Omega$.
- If $B \in \mathcal{F}$, then either $A \subset B$, in which case $A \cap B^c = \varnothing$, of $A \cap B = \varnothing$, in which case $A \subset B^c$; either way, we have $B^c \in \mathcal{F}$.
- Finally, take $B_1, B_2, \ldots \in \mathcal{F}$ and let us show that $\cup B_n \in \mathcal{F}$. In case we have $A \cap B_n = \varnothing$ for every n, then $A \cap (\cup B_n) = \varnothing$, so $\cup B_n \in \mathcal{F}$. In case we have $A \subset B_N$ for some N, then we also have $A \subset \cup B_n$, so again $\cup B_n \in \mathcal{F}$.

Now, by assumption we have $\mathcal{E} \subset \mathcal{F}$, so $\sigma(\mathcal{E}) \subset \sigma(\mathcal{F}) = \mathcal{F}$, where the last inequality holds because $\mathcal{F}$ is a σ-algebra. This means that every set $B \in \sigma(\mathcal{E})$ satisfies the property that defines $\mathcal{F}$, that is, either $A \cap B = \varnothing$ or $A \subset B$; in either case, we must have $B \neq A_0$.

(b) Let Ω be a set and $\mu^\star$ be an outer measure on Ω. Suppose $A \subset \Omega$ is $\mu^\star$-measurable. Show that, for any $B \subset \Omega$ with $\mu^\star(B) < \infty$, we have

$$\mu^\star(A \cup B) = \mu^\star(A) + \mu^\star(B) - \mu^\star(A \cap B).$$

Solution. Since A is $\mu^\star$-measurable, we have

$$\mu^\star(Z) = \mu^\star(Z \cap A) + \mu^\star(Z \cap A^c) \quad \forall Z \subset \Omega.$$

Applying this to $Z = B$ and to $Z = A \cup B$ respectively yields:

$$\mu^\star(A \cup B) = \mu^\star(A) + \mu^\star(A^c \cap B);$$
$$\mu^\star(B) = \mu^\star(A \cap B) + \mu^\star(A^c \cap B) \quad \Longrightarrow \quad \mu^\star(A^c \cap B) = \mu^\star(B) - \mu^\star(A \cap B);$$

note that in the implication in the second line we used the fact that $\mu^\star(A \cap B) \leq \mu^\star(B) < \infty$. Putting these equalities together gives the desired result.

2. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set (that is, $E \in \mathcal{M}$). Prove that, if $m(E) > 0$, then for every $\varepsilon > 0$ there exists an interval $[a, b]$ such that $m(E \cap [a, b]) > (1-\varepsilon) \cdot (b-a)$.

Solution. Suppose to the contrary that there exists $\varepsilon > 0$ such that, for every interval $[a, b]$, we have $m(E \cap [a, b]) \leq (1 - \varepsilon) \cdot (b - a)$. Fix $\delta > 0$ small enough that

$$(1 - \varepsilon) \cdot (m(E) + \delta) < m(E).$$

By the definition of Lebesgue outer measure, there exist a sequence of intervals $[a_n, b_n]$, $n \in \mathbb{N}$, such that $E \subset \cup [a_n, b_n]$ and

$$m(E) \leq \sum_n (b_n - a_n) \leq m(E) + \delta.$$

But then,

$$E \subset \cup [a_n, b_n] \implies E \subset \cup([a_n, b_n] \cap E)$$
$$\implies m(E) \le \sum_n m([a_n, b_n] \cap E) \le (1-\varepsilon) \sum_n (b_n - a_n) \le (1-\varepsilon)(m(E)+\delta) < m(E).$$

Hence it cannot be the case that $m(E \cap [a,b]) \le (1-\varepsilon) \cdot (b-a)$ for every $[a,b]$.

3. In both the following items, $\mathbb{R}$ is endowed with the Borel σ-algebra.

   (a) Let $(\Omega, \mathcal{A})$ be a measurable space, and let $A_n \in \mathcal{A}$, $n \in \mathbb{N}$. Define $f : \Omega \to \bar{\mathbb{R}}$ by

   $$f(\omega) = \inf\{n : \omega \in A_m \text{ for all } m \ge n\}, \quad \omega \in \Omega$$

   (we adopt the convention that $\inf \varnothing = \infty$). Prove that f is measurable.

   Solution. For every $x \in \bar{\mathbb{R}}$, we have

   $$\{\omega : f(\omega) \le x\} = \bigcup_{n \in \mathbb{N}: n \le x} \{\omega : f(\omega) \le n\} = \bigcup_{\substack{n \in \mathbb{N}:\\ n \le x}} \bigcap_{\substack{m \in \mathbb{N}:\\ m \ge n}} A_m \in \mathcal{A}.$$

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be right continuous. Prove that f is measurable.

   Solution. For each $n \in \mathbb{N}$, define

   $$f_n = \sum_{k \in \mathbb{Z}} f((k+1)2^{-n}) \cdot \mathbb{1}_{(k2^{-n},(k+1)2^{-n}]}.$$

   Since f_n is a sum of indicator functions of measurable sets, it is measurable. We will prove that $f_n \to f$ pointwise; from this it will follow that f is measurable. Fix $x \in \mathbb{R}$ and $\varepsilon > 0$. Since f is right continuous, there exists $\delta > 0$ such that, if $y \in [x, x+\delta]$, then $|f(y) - f(x)| < \varepsilon$. Now take $n \in \mathbb{N}$ with $2^{-n} < \delta$. Let y_n be the smallest number of the form $k2^{-n}$ with $k \in \mathbb{Z}$ that is larger than x. Then, $f_n(x) = f(y_n)$, so $|f_n(x) - f(x)| = |f(y_n) - f(x)| < \varepsilon$ since $y_n \in [x, x+\delta]$.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space.

   (a) Let $f : \Omega \to \bar{\mathbb{R}}$ be an integrable function satisfying $\int_E f \, d\mu \ge 0$ for all $E \in \mathcal{A}$. Prove that $f \ge 0$ almost everywhere.

   Solution. Assume to the contrary that $\mu(\{\omega : f(\omega) < 0\}) > 0$. Then, since

   $$\mu(\{\omega : f(\omega) < 0\}) = \lim_{n \to \infty} \mu(\{\omega : f(\omega) < -1/n\}),$$

   there must exist n such that $\mu(\{\omega : f(\omega) < -1/n\}) > 0$. But then,

   $$\int_{\{\omega : f(\omega) < -1/n\}} f \, d\mu \le -\frac{1}{n} \cdot \mu(\{\omega : f(\omega) < -1/n\}) < 0.$$

   (b) Let $g : \Omega \to \bar{\mathbb{R}}$ be an integrable function. Prove that

   $$\lim_{n \to \infty} n \cdot \mu(\{\omega : g(\omega) > n\}) = 0.$$

Solution. First note that

$$\infty \cdot \mu(\{\omega : |g(\omega)| = \infty\}) = \int_{\{\omega:|g(\omega)|=\infty\}} |g|\ d\mu \le \int_\Omega |g|\ d\mu < \infty,$$

so we have $\mu(\{\omega : |g(\omega)| = \infty\}) = 0$. Next, let $g_n = g \cdot \mathbb{1}_{\{\omega:g(\omega)>n\}}$. We have the pointwise convergence $g_n \to \infty \cdot \mathbb{1}_{\{\omega:g(\omega)=\infty\}}$, and this convergence is dominated by g, which is integrable. Hence, using the Dominated Convergence Theorem,

$$\begin{aligned} n \cdot \mu(\{\omega : g(\omega) > n\}) &= \int n \cdot \mathbb{1}_{\{\omega:g(\omega)>n\}}\ d\mu \\ &\le \int g_n\ d\mu \xrightarrow{n\to\infty} \infty \cdot \mu(\{\omega : g(\omega) = \infty\}) = 0. \end{aligned}$$

5. In this exercise, we consider the set $\Omega = (0, \infty)$ with the Borel σ-algebra and Lebesgue measure (these are just the restrictions to $(0, \infty)$ of the Borel σ-algebra and Lebesgue measure of $\mathbb{R}$).
   Let $p > 1$ and let q be the conjugate exponent of p, that is, $p + q = pq$. Assume $f \in L^p((0, \infty))$.

   (a) Show that, for every $x > 0$, $f \cdot \mathbb{1}_{(0,x)} \in L^1((0, \infty))$.

   Solution. We have $|f(t)| \le 1 + |f(t)|^p$ for all $t > 0$, so

   $$\int_0^\infty |f \cdot \mathbb{1}_{(0,x)}|\ dt = \int_0^x |f(t)|\ dt \le \int_0^x (1 + |f(t)|^p)\ dt \le x + \int_0^\infty |f(t)|^p\ dt < \infty.$$

   (b) Prove that, for any $\alpha \in (0, 1/q)$ and $x > 0$,

   $$\left|\int_0^x f(t)\ dt\right| \le \frac{x^{\frac{1}{q}-\alpha}}{(1-\alpha q)^{\frac{1}{q}}} \left(\int_0^x t^{\alpha p} \cdot |f(t)|^p\ dt\right)^{\frac{1}{p}}.$$

   Hint. Write $f(t) = t^{-\alpha} \cdot t^\alpha \cdot f(t)$ and use Hölder's inequality (make sure to verify that the assumptions for the inequality are satisfied!)

   Solution. Given $x > 0$, define

   $$g(t) = t^{-\alpha} \cdot \mathbb{1}_{(0,x)}(t), \qquad h(t) = t^\alpha \cdot f(t) \cdot \mathbb{1}_{(0,x)}(t),$$

   so that $\int_0^x f(t)\ dt = \int_0^\infty g(t) \cdot h(t)\ dt$. Note that

   $$\int_0^\infty |g(t)|^q\ dt = \int_0^x t^{-\alpha q}\ dt = \frac{x^{1-\alpha q}}{1-\alpha q} < \infty$$

   since $\alpha q < 1$, so $g \in L^q((0, \infty))$. Also,

   $$\int_0^\infty |h(t)|^p\ dt = \int_0^x t^{\alpha p} \cdot |f(t)|^p\ dt \le x^{\alpha p} \cdot \int_0^\infty |f(t)|^p\ dt < \infty,$$

   so $h \in L^p((0, \infty))$. We can thus use Hölder's inequality to obtain:

   $$\begin{aligned} \left|\int_0^x f(t)\ dt\right| \le \int_0^x |f(t)|\ dt &= \int_0^\infty |g(t)| \cdot |h(t)|\ dt \\ &\le \left(\int_0^\infty |g(t)|^q\ dt\right)^{\frac{1}{q}} \cdot \left(\int_0^\infty |h(t)|^p\ dt\right)^{\frac{1}{p}} \\ &= \frac{x^{\frac{1}{q}-\alpha}}{(1-\alpha q)^{\frac{1}{q}}} \cdot \left(\int_0^x t^{\alpha p} \cdot |f(t)|^p\ dt\right)^{\frac{1}{p}}. \end{aligned}$$

(c) Define, for $x > 0$,

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt.$$

Prove that $F \in L^p((0,\infty))$. *Hint.* You will need part (b) and the Fubini-Tonelli theorem. Use the fact that $p - \frac{p}{q} = 1$.

Solution. By part (b),

$$\begin{aligned} |F(x)|^p = \left| \frac{1}{x} \cdot \int_0^x f(t)\, dt \right|^p &\le \frac{x^{p(\frac{1}{q}-\alpha-1)}}{(1-\alpha q)^{\frac{p}{q}}} \cdot \int_0^x t^{\alpha p} \cdot |f(t)|^p\, dt \\ &= \frac{x^{-\alpha p - 1}}{(1-\alpha q)^{\frac{p}{q}}} \cdot \int_0^x t^{\alpha p} \cdot |f(t)|^p\, dt, \end{aligned}$$

where the equality follows from $p - \frac{p}{q} = 1$. Now we have

$$\int_0^\infty |F(x)|^p\, dx \le \int_0^\infty \frac{x^{-\alpha p - 1}}{(1-\alpha q)^{\frac{p}{q}}} \cdot \int_0^x t^{\alpha p} \cdot |f(t)|^p\, dt\, dx.$$

Note that the function

$$(x,t) \mapsto \frac{x^{-\alpha p - 1}}{(1-\alpha q)^{\frac{p}{q}}} \cdot t^{\alpha p} \cdot |f(t)|^p \cdot \mathbb{1}_{\{t \le x\}}$$

is non-negative and measurable with respect with the product Borel σ-algebra on $(0,\infty)$ (since it is a product of measurable functions). Hence, by the Fubini-Tonelli theorem,

$$\begin{aligned} &\int_0^\infty \frac{x^{-\alpha p - 1}}{(1-\alpha q)^{\frac{p}{q}}} \cdot \int_0^x t^{\alpha p} \cdot |f(t)|^p\, dt\, dx \\ &= \int_0^\infty \int_t^\infty \frac{x^{-\alpha p - 1}}{(1-\alpha q)^{\frac{p}{q}}} \cdot t^{\alpha p} \cdot |f(t)|^p\, dx\, dt \\ &= \frac{1}{(1-\alpha q)^{\frac{p}{q}}} \int_0^\infty t^{\alpha p} \cdot |f(t)|^p \int_t^\infty x^{-\alpha p - 1}\, dx\, dt \\ &= \frac{1}{(-\alpha q + 1)^{p/q}} \cdot \frac{1}{\alpha p} \cdot \int_0^\infty t^{\alpha p} \cdot |f(t)|^p \cdot t^{-\alpha p} dt = \frac{1}{(-\alpha q + 1)^{p/q}} \cdot \frac{1}{\alpha p} \cdot \|f\|_p^p. \end{aligned}$$

We have thus shown that $\int_0^\infty |F(x)|^p\, dx < \infty$, so $F \in L^p((0,\infty))$.